Extended Sensitivity Analysis for Heterogeneous Unmeasured Confounding with an Application to Sibling Studies of Returns to Education

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Heterogeneous Patterns of Confounding

- A **sensitivity analysis** assesses how robust causal conclusions from an observational study are to departures from random assignment due to unmeasured confounding.
- In a stratified study, the **sensitivity value** is the smallest bound on the maximal bias present in any stratum, while allowing for *arbitrarily unfavorable patterns of confounding*, such that the qualitative conclusions of the study change.
- If the investigator suspects hidden confounding is **present but heterogeneous** across strata *the least favorable patterns of confounding may be very unlikely*, leading the sensitivity analysis to be **overly pessimistic**.

Goal of the Paper: Extend sensitivity analysis to place bounds on both maximal and typical biases present in strata, to account for the presence of heterogeneous confounding.

Example: Returns to Education

Ashenfelter and Rouse (1998):

- Collected survey data on 340 monozygotic twin pairs from the Twinsburg Twins Festival in Twinsburg, Ohio over three summers (1991-1993).
- $I = 40$ pairs where one twin had at least two years of college education ($Z_{ij} = 1$) and the other no more than a HS degree ($Z_{ij'} = 0$).
- Included twin pairs where both twins held a job at the time of survey collection; log wage data reflecting the 1989-1993 period ($R_{ij} = Z_{ij} r_{Tij} + (1 - Z_{ij}) r_{Cij}$).
- Retrospective survey $\implies$ no measures of baseline ability, such as IQ recorded earlier in life (**a potential unobserved confounder** u_{ij}).

Naive analysis assuming no unmeasured confounding:

- p-value ≈ 0.0001 (Fisher's Sharp Null – $H_0 : \mathbf{r}_T = \mathbf{r}_C$)
- 95% CI: $[16\%, 43\%]$

A Model for Unmeasured Confounding

- Assuming **no unmeasured confounding** between twins, treatment assignment in a twin study looks like a randomized experiment:

$$\mathbb{P}(\mathbf{Z} = \mathbf{z} | \, \mathbf{z} \in \Omega) = 1/2^I, \quad \Omega = \{\mathbf{z} : z_{i1} + z_{i2} = 1, \forall\, i = 1, \ldots, I\}.$$

- In the **presence of unmeasured confounding** ($u_{i1} \neq u_{i2}$), distribution of treatment allocations is **unknown**:

$$\mathbb{P}_{\boldsymbol{\pi}}(\mathbf{Z} = \mathbf{z} | \, \mathbf{z} \in \Omega) = \prod_{i=1}^{I} \pi_{i1}^{Z_{i1}} (1 - \pi_{i1})^{1 - Z_{i1}}$$

because $\pi_{ij} = \mathbb{P}(Z_{ij} = 1 \,|\, \mathbf{Z} \in \Omega)$ are **unobserved**.

- Departures from randomization parameterized by $\Gamma > 1$, a **bound on the maximal odds of treatment assignment** in any twin pair:

$$\pi_i^*/(1 - \pi_i^*) \leq \Gamma, \quad \pi_i^* = \max\{\pi_{i1}, \pi_{i2}\}, \quad \forall\, i = 1, \ldots, I.$$

- **Conventional Sensitivity Analysis under this model:** For a test statistic $t = \mathbf{Z}^T \mathbf{q}(\mathbf{R})$ and given maximal bias bound Γ, find the pattern of $\boldsymbol{\pi}$ that maximizes p-value $P(\boldsymbol{\pi})$ under reference distribution $\mathbb{P}_{\boldsymbol{\pi}}$,

$$P_\Gamma^* = \max_{\boldsymbol{\pi} \in \mathcal{P}_\Gamma} P(\boldsymbol{\pi})$$

where $\mathcal{P}_\Gamma = \{\boldsymbol{\pi} : 1/2 \leq \pi_i^* \leq \Gamma/(1 + \Gamma), \forall\, i = 1, \ldots, I\}$

Conventional Sensitivity Analysis of Example

- **Worst case pattern of confounding:** Amounts to assigning twin with higher hourly wage the largest allowable probability of attending college $\implies$ **average and maximal π_i^* coincide.**
- **Sensitivity value:** $\Gamma^* \approx 2.36$ where $\Gamma^* = \inf\{\Gamma : \, P_\Gamma^* \geq 0.05/2\}$.

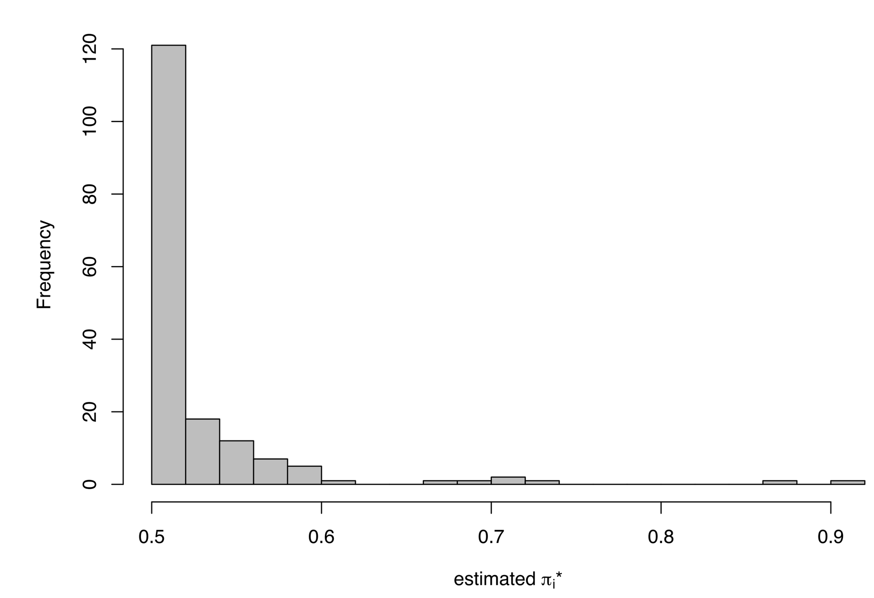
Is 2.36 a plausible bound on the maximal bias among twins due to unmeasured confounding?

- **Example: Ability Bias** – *Could disparities in ability between twins in our study increase the odds that one of the twins went to college by more than a factor of 2.36?*

Cross-Study Calibration

A partial solution to the above open question: Calibrate Γ^* to an estimate of maximal bias due to disparities in ability from a **calibration study** that has comparable design and information on baseline IQ [using calibration methods from Hsu and Small (2013)]. For example,

Estimated π_i^* from 171 same-sex sibling pairs in the Wisconsin Longitudinal Study (WLS) where $Z_{i1} + Z_{i2} = 1$:



- $\max \pi_i^*/(1 - \max \pi_i^*) = 9.3$ and $\bar{\pi}^*/(1 - \bar{\pi}^*) = 1.1 \implies$ mostly modest biases, some moderate and few very large biases.
- $\Gamma^* < 9.3 \implies$ Ashenfelter study **likely sensitive** to plausible levels of ability bias but also suggests *maximal and typical biases due to IQ disparities are quite different*.

Extended Sensitivity Analysis

Our Contribution: Adapt the conventional sensitivity analysis to provide a less pessimistic view of the study's robustness to hidden bias when heterogeneous confounding leads to differing bounds on maximal and typical bias.

Extended Sensitivity Analysis (ESA): A two-parameter sensitivity analysis that simultaneously bounds the maximal and typical biases by Γ and $\bar{\Gamma}$, respectively:

$$P_{\Gamma,\bar{\Gamma}}^* = \max_{\boldsymbol{\pi} \in \mathcal{P}_{\Gamma,\bar{\Gamma}}} P(\boldsymbol{\pi})$$

where $\mathcal{P}_{\Gamma,\bar{\Gamma}}$ is the set of $\boldsymbol{\pi}$ that respect both bounds on maximal and typical biases.

Properties:

- Returns a curve of $(\Gamma, \bar{\Gamma})$ changepoints, analagous to Γ^*.
- If desired, bias parameters can be calibrated to population-level bias information (important in cross-study calibration).
- Can recover conventional sensitivity analysis when $\bar{\Gamma} = \Gamma$.
- Computationally feasible – can be expressed as a standard QP [Fogarty and Small (2016)].

Superpopulation Model for Paired Studies

- Sample a treated subject from an infinite population of treated subjects and record $X_{i1} = x_i$ then sample a control subject from an infinite population of controls conditional on $X_{i2} = x_i$.
- Under this model, π_{ij} is a realization of random variable Π_{ij} $\implies$ π_i^* a realization of random variable $\Pi_i^* = \max\{\Pi_{i1}, \Pi_{i2}\}$.

Why assume a population when inferential framework is conditional on sample?

Why Population-Level Bias Calibration?

Bias estimates from calibration sample are relevant to study sample only insofar as the two samples believably came from some common superpopulation!

A difficulty with constructing a set $\mathcal{P}_{\Gamma,\bar{\Gamma}}$ in terms of bounds on population-level bias:

- Although $1/2 \leq \Pi_i^* \leq \Gamma/(1+\Gamma) \implies 1/2 \leq \max \pi_i^* \leq \Gamma/(1 + \Gamma)$, the same correspondence does not hold for typical biases,

$$1/2 \leq \mathbb{E}[\Pi_i^*] \leq \bar{\Gamma}/(1 + \bar{\Gamma}) \;\not\Longrightarrow\; 1/2 \leq \bar{\pi}_i^* \leq \bar{\Gamma}/(1 + \bar{\Gamma}).$$

Implication for constructing $\mathcal{P}_{\Gamma,\bar{\Gamma}}$: A deterministic set $\mathcal{P}_{\Gamma,\bar{\Gamma}}$ respecting population-level bias bounds would allow for an allocation $\boldsymbol{\pi}$ such that $\bar{\pi}^*$ and $\max \pi_i^*$ coincide, however unlikely $\implies$ **back in the conventional sensitivity analysis.**

How can we remedy this?

Making $\mathcal{P}_{\Gamma,\bar{\Gamma}}$ Stochastic

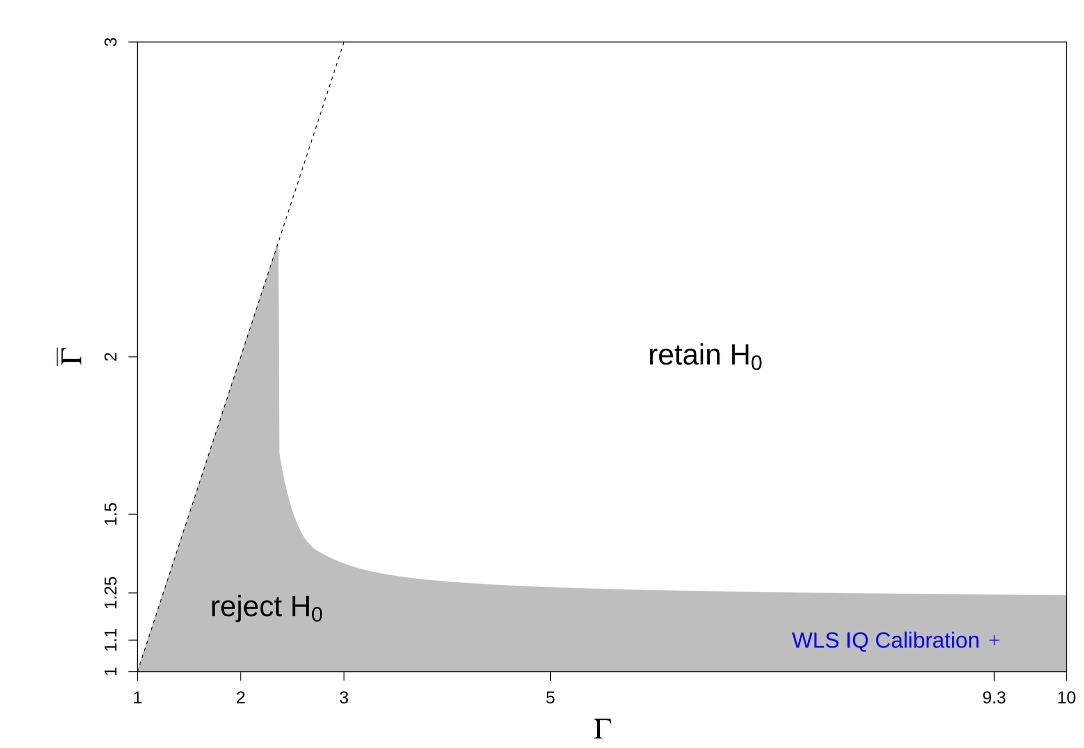
A strategy from Berger and Boos (1994): Maximize p-value over $1 - \beta$ confidence set for a nuisance parameter and add β to get a valid p-value for tests based on non-pivotal quantities.

Modification for ESA: $\boldsymbol{\pi}$ takes the place of the nuisance parameter, need $\boldsymbol{\Pi} \in \mathcal{P}_{\Gamma,\bar{\Gamma}}$ with probability at least $1 - \beta$, and β is added to the maximal p-value to account for stochastic nature of set.

Ensuring $\mathbb{P}(\boldsymbol{\Pi} \in \mathcal{P}_{\Gamma,\bar{\Gamma}}) \geq 1 - \beta$: Can use fact that $\bar{\Pi}^*$ is a sample average, conservative variance estimate for Π_i^* that depends only on $\bar{\Gamma}$ and Γ (Bhatia-Davis Inequality), and apply CLT.

Calibrated Sensitivity Curve

Rather than a single sensitivity parameter, ESA returns a **sensitivity curve** of $(\Gamma, \bar{\Gamma})$ below which we cannot reject H_0, calibrated to estimates of **both** typical and maximal biases due to disparities in IQ:



Takeaway: Unlike the conventional analysis, the calibrated ESA suggests that the Ashenfelter study **is not sensitive** to plausible patterns of ability bias!

Special Cases:

- At (Γ, Γ) we recover the conventional sensitivity analysis $\implies \Gamma^* \approx 2.36$.
- If no plausible bound on maximal bias, taking $\lim_{\Gamma_n \to \infty} (\Gamma_n, \bar{\Gamma})$ recovers single-parameter sensitivity analysis bounding only the typical bias $\implies \bar{\Gamma}^* \approx 1.22$.

Concluding Remarks

- In summary, the extended sensitivity analysis generalizes the conventional sensitivity analysis by allowing the researcher to place bounds on both the maximal and typical bias.
- As illustrated with the Ashenfelter study, this will generally result in less conservative conclusions when there is reason to believe the bounds on maximal and typical bias do not coincide.

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